

ON THE WELL-POSEDNESS OF RELATIVISTIC VISCOUS FLUIDS WITH NON-ZERO VORTICITY

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ABSTRACT. We study the problem of coupling Einstein's equations to a relativistic and physically well-motivated modification of the Navier-Stokes equations. Under a technical condition for the vorticity, we prove existence and uniqueness in a suitable Gevrey class if the fluid's dynamic velocity has vanishing divergence, and show that the solutions enjoy the finite propagation speed property.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS.

This paper improves the results of [13], where the formulation of relativistic viscous fluids has been investigated by the second author. There, following Lichnerowicz [52], a physically well-motivated relativistic version of the Navier-Stokes equations has been proposed, and well-posedness, in Gevrey spaces, of the corresponding Einstein-Navier-Stokes system established under the assumption that the fluid is incompressible (in a relativistic sense, see below) and irrotational. The goal of the present work is to remove the latter hypothesis, replacing it by a restriction on the initial data that allows the vorticity to be non-zero.

Finding the correct way of incorporating viscosity into General Relativity is a longstanding problem¹ [52, 60, 79], one that has recently attracted attention due to its importance in the study of heavily dense objects (as neutron stars), and models of the early universe. See, for instance, [12, 15, 18, 19, 20, 21, 37, 38, 48, 57, 59, 65, 66, 67, 68, 70, 73, 74, 78] and references therein. A thorough and more up-to-date discussion, including details on the First and Second Order Theories mentioned below, can be found in [70].

The main difficulty in formulating a theory of relativistic viscous fluids seems to stem from the absence of a variational formulation for the classical, non-relativistic, Navier-Stokes equations (see, however, [24, 80] for formalisms that allow for a variational principle in some more general sense). Lacking such a formulation, one does not have a canonical way of determining what the stress-energy tensor $T_{\alpha\beta}$ ought to be in the context of General Relativity. Different proposals have been made in this regard, but they have all led to either ill-posed equations, or to equations that imply the existence of superluminal signals. This approach, where one couples Einstein's equations to the Navier-Stokes equations via the introduction of a suitable stress-energy tensor is known as (relativistic) Standard Irreversible Thermodynamics or First Order Theories. We remark that it consists in the traditional approach of coupling gravity to matter, one that has been successful for almost all other matter fields [8, 36].

The failure of First Order Theories in producing a consistent theory of relativistic fluids led researchers to devise a different approach, known as (relativistic) Rational Extended Irreversible Thermodynamics, or Second Order Theories, or yet Divergence-type formulation of Extended Irreversible Thermodynamics [47, 63, 70].

In such theories, one extends the space of variables of the model, and the resulting equations are of hyperbolic character in several important situations of physical interest, leading to equations that are well-posed, with disturbances propagating at finite speed. It is not at all clear, however, that the equations remain hyperbolic under all physically realistic scenarios. In fact, Rezzolla and Zanotti conclude their detailed discussion of relativistic viscous fluids pointing out that “the construction of a formulation that is cast in a divergence-type is not, *per se*, sufficient to guarantee hyperbolicity” [70]. Furthermore, the plethora of models that comes out of the extended thermodynamic approach suggests that it entails many *ad-hoc* features, in sharp contrast to the usually unique way of coupling gravity to matter via the introduction of the stress-energy tensor of matter fields (when the latter is uniquely determined by a variational characterization).

These considerations suggest that it is worthwhile to take a fresh look at the question of whether there is a correct stress-energy tensor $T_{\alpha\beta}$ that describes relativistic viscous fluids, and that can be coupled to gravity in the traditional way, i.e., as in the Standard Irreversible Thermodynamics approach (see also the discussion in section 2.3). This idea is reinforced by the fact that recent numerical advances in the modeling of rapidly rotating stars with shear viscosity employ the first order formalism [21].

¹It is interesting to notice that even the correct formulation of the non-relativistic Navier-Stokes equations on a general Riemannian manifold does not seem to present itself in a natural and obvious way, see [5, 22, 76].

Consider the following stress-energy tensor for a viscous fluid:

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - pg_{\alpha\beta} + \kappa\pi_{\alpha\beta}\nabla_\mu C^\mu + \vartheta\pi_\alpha^\rho\pi_\beta^\mu(\nabla_\rho C_\mu + \nabla_\mu C_\rho), \quad (1.1)$$

where p and ϱ are respectively the pressure and density of the fluid, u is its four-velocity, the bulk viscosity κ and the shear viscosity ϑ are non-positive constants², g is a Lorentzian metric³ and $\pi_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$. p and ϱ are related by an equation known as equation of state, the choice of which depends on the nature of fluid. C is known as the dynamic velocity (also called the current of the fluid), and it is related to u by

$$C_\alpha = Fu_\alpha, \quad (1.2)$$

where F is the so-called index of the fluid. It is defined as

$$F = 1 + \epsilon + \frac{p}{r}, \quad (1.3)$$

where $\epsilon \geq 0$ is the specific internal energy and $r \geq 0$ is the rest mass density [33]. The density ϱ is related to the internal energy and the rest mass by

$$\varrho = r(1 + \epsilon),$$

so that

$$rF = \varrho + p. \quad (1.4)$$

The index of the fluid, F , and the dynamic velocity, C , have been introduced by Lichnerowicz in his study of relativistic inviscid fluids [52, 53, 54, 55].

Lichnerowicz was also the first one to write down the stress-energy tensor (1.1) [52], except that it contained an extra term of the form $\vartheta\pi_{\alpha\beta}u^\mu\partial_\mu F$. This extra term was pointed out by Lichnerowicz himself and later by Pichon [69], to lead to an indetermination in the computation of the pressure. Pichon proposed subtracting this term, which leads to (1.1). See [52, 69] for more background on (1.1). The reader should notice that (1.1) reduces to the stress-energy tensor of an ideal, i.e., inviscid, fluid when $\kappa = \vartheta = 0$. Indeed, this is just one of several natural requirements that one would impose when looking for an appropriate definition of a stress-energy tensor for a relativistic fluid with viscosity, see [13]. We point out that Choquet-Bruhat has also proposed a stress-energy tensor similar to (1.1) [8]. Her proposal does not include the projection terms $\pi_{\alpha\beta}$, and the viscous terms are, therefore, linear in the velocity. We remark that yet another proposal for a viscous relativistic stress-energy tensor appears in [28].

Next, recall the first law of thermodynamics⁴

$$\theta ds = d\epsilon + pdv \quad (1.5)$$

where θ is the absolute temperature, s the specific entropy, and v the specific volume. We have $v = \frac{1}{r}$ [33], so by (1.3), (1.5) can be written as

$$\theta ds = dF - \frac{1}{r}dp. \quad (1.6)$$

²The coefficients of bulk and shear viscosity have a definite sign, the choice of which depends on conventions. Sometimes $T_{\alpha\beta}$ is written with a shear term $-\vartheta\pi_\alpha^\rho\pi_\beta^\mu(\nabla_\rho C_\mu + \nabla_\mu C_\rho)$, $\vartheta > 0$, which corresponds to having $\vartheta < 0$ in our formulation. While such sign differences are important when one explicitly computes the values of physical observables, for the results here presented all that matters is that $\vartheta \neq 0$. In physically realistic models, it is also the case that ϑ is not a constant, but a smooth function of the thermodynamic variables. Our result can be extended to this case with minor changes in the proof, provided that ϑ never vanishes, but we do not include this here for brevity.

³Our convention for the metric is $(+ - - -)$.

⁴See, for instance, [14, 33] for a review of the thermodynamic properties of relativistic fluids.

A fluid with stress-energy tensor (1.1) is said to be incompressible if

$$\nabla_\mu C^\mu = 0. \quad (1.7)$$

Remark 1.1. Here we follow the literature (e.g., [8, 55]) and call incompressible a fluid satisfying (1.7). We stress, however, that this terminology is based more on an analogy with Newtonian physics (where incompressible fluids are characterized by vanishing divergence) than on actual physical properties of fluids, in that (1.7) does not imply incompressibility in the exact sense of the word. Pseudo-incompressible would probably be a better terminology, but it is not clear if adopting a different terminology than what is used in the literature would not cause more confusion.

Then, by (1.4), $T_{\alpha\beta}$ becomes

$$T_{\alpha\beta} = rFu_\alpha u_\beta - pg_{\alpha\beta} + \vartheta\pi_\alpha^\rho\pi_\beta^\mu(\nabla_\rho C_\mu + \nabla_\mu C_\rho). \quad (1.8)$$

Moreover, because $\pi_\beta^\mu u_\mu = 0$, we can rewrite (1.8) as

$$T_{\alpha\beta} = rFu_\alpha u_\beta - pg_{\alpha\beta} + F\vartheta\pi_\alpha^\rho\pi_\beta^\mu(\nabla_\rho u_\mu + \nabla_\mu u_\rho). \quad (1.9)$$

Finally, we define the vorticity tensor by

$$\Omega_{\alpha\beta} = \nabla_\alpha C_\beta - \nabla_\beta C_\alpha \equiv \partial_\alpha C_\beta - \partial_\beta C_\alpha. \quad (1.10)$$

A fluid is called irrotational if $\Omega = 0$. Notice that Ω is anti-symmetric, so it has only six independent components.

We follow the standard approach of assuming that only two of the thermodynamic quantities are independent with the question of which ones left as a matter of choice. The other quantities are then determined by the first law of thermodynamics and an equation of state. The equation of state depends on the nature of the fluid, and physically, the relations between the thermodynamic quantities should be invertible⁵. Here we shall assume that r and s are independent and postulate an equation of state of the form

$$\varrho = \mathcal{P}(r, s). \quad (1.11)$$

It follows that $p = p(r, s)$, $\theta = \theta(r, s)$, $\epsilon = \epsilon(r, s)$, and $F = F(r, s)$ are known if r and s are. We note that later on it will be more convenient to treat s and F as independent variables. Then the equation of state will be given by $r = r(F, s)$.

On physical grounds, one has that $F > 0$. This allows to restrict to positive values when treating F as an independent variable. In this situation, the following condition will be assumed to hold:

$$\frac{\partial r}{\partial F} \geq \frac{r}{F}, \quad (1.12)$$

in particular $\frac{\partial r}{\partial F} > 0$ if $r > 0$. Condition (1.12) expresses the statement that sound waves in an ideal fluid travel at most at the speed of light. This condition has to be satisfied if we want to recover the stress-energy tensor of an ideal fluid when $\kappa = \vartheta = 0$ [55]. We suppose that the equation of state is such that the temperature satisfies

$$\begin{aligned} \theta(r, s) &> 0 \text{ if } r > 0, s \geq 0, \\ \theta(F, s) &> 0 \text{ if } s \geq 0, F > 0, \end{aligned} \quad (1.13)$$

expressing the positivity of the temperature regardless of the choice of independent variables.

⁵Upon making such assumptions, we are restricting ourselves to fluids in a single phase and ruling out the possibility of phase transitions.

The full system of equations derived from coupling Einstein's equations to (1.8) is rather complicated, see [13]. Thus we consider, besides incompressibility, one further simplifying assumption, namely, we investigate the sub-class of solutions for which the vorticity evolves according to

$$\mathcal{L}_C \Omega_{\alpha\beta} + qu^\mu \nabla_\mu \nabla_\alpha C_\beta - qu^\mu \nabla_\mu \nabla_\beta C_\alpha + \partial_\alpha(\theta F) \partial_\beta s - \partial_\beta(\theta F) \partial_\alpha s = \mathcal{F}_\vartheta. \quad (1.14)$$

Here, \mathcal{L}_C denotes the Lie derivative in the direction of C , \mathcal{F}_ϑ is a smooth function of Ω , g and its derivatives up to second order, C and u and their derivatives up to first order. q is a constant, and \mathcal{F}_ϑ and q may also depend on the parameter ϑ . \mathcal{F}_ϑ and q dictate, to a certain extent, which quantities are considered relevant in some particular model, and, therefore, are chosen according to the problem one wishes to study (see section 2.4). We discuss the restrictions imposed by (1.14) in section 2.1. It should be noticed that one must have $q = 0$ and that \mathcal{F}_ϑ cannot be chosen freely if $\vartheta = 0$ (although $\vartheta = 0$ will not be treated here).

The starting point is the following system of equations: Einstein's equations coupled to (1.1) and supplemented by (1.7) and (1.14), reading

$$\begin{cases} R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \mathcal{K} T_{\alpha\beta}, & (1.15a) \\ \nabla^\alpha T_{\alpha\beta} = 0, & (1.15b) \\ \nabla_\alpha(r u^\alpha) = 0, & (1.15c) \\ \nabla_\mu C^\mu = 0, & (1.15d) \\ \mathcal{L}_C \Omega_{\alpha\beta} + qu^\mu \nabla_\mu \nabla_\alpha C_\beta - qu^\mu \nabla_\mu \nabla_\beta C_\alpha = \tilde{\mathcal{F}}_\vartheta, & (1.15e) \\ u^\alpha u_\alpha = 1. & (1.15f) \end{cases}$$

where $R_{\alpha\beta}$ and R are the Ricci and scalar curvature of the metric g , \mathcal{K} is a coupling constant, and Λ is the cosmological constant⁶. We recall that (1.15b) is in fact a consequence of (1.15a) in view of the Bianchi identities, but it is customary to list it along with the other equations. In the sequel we set \mathcal{K} to 1. The equation (1.15c) says the mass is locally conserved along the flow lines, and (1.15f) is the standard normalization condition on the velocity of a relativistic fluid. In general, without (1.15c), the motion of the fluid is underdetermined. Equation (1.15e) is simply (1.14), with $\tilde{\mathcal{F}}_\vartheta$ defined in the obvious fashion. A useful consequence of (1.15f) often used in computations is

$$u^\alpha \nabla_\beta u_\alpha = 0. \quad (1.16)$$

The unknowns are the metric g , the fluid velocity u , the specific entropy s , and the rest mass density r , where s and r are non-negative real valued functions. We suppose that we are also given a smooth function $\mathcal{P} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ that gives the equation of state (1.11), with the other thermodynamic quantities then given as functions of s and r as discussed above.

Definition 1.2. System (1.15) with $T_{\alpha\beta}$ given by (1.1) will be called the incompressible Einstein-Navier-Stokes system.

Remark 1.3. Here we recall once more that the terminology “incompressible fluid” is a bit misleading, see remark 1.1.

Assumption. We shall assume for the rest of the text that $\vartheta \neq 0$.

An initial data set for the Einstein-Navier-Stokes system consists of the following:

- a three-dimensional manifold Σ ,
- a Riemannian metric g_0 (with our conventions this metric is negative definite)

⁶Our results do hold irrespective of the value of Λ .

- a symmetric two-tensor κ ,
- a real valued non-negative function s_0 ,
- a real valued non-negative function r_0 ,
- a vector field v .

The last five quantities are defined on Σ . As it is well-known, these data must satisfy the constraint equations. In a coordinate system with ∂_0 transversal and ∂_i , $i = 1, 2, 3$, tangent to Σ the constraint equations are given by

$$S_{\alpha 0} = T_{\alpha 0}, \quad (1.17)$$

where $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta}$ is the Einstein tensor. In our case, it is not enough that the initial data satisfies (1.17). We also need the compatibility conditions obtained upon restriction of (1.15e) to the initial hypersurface, since initial data for Ω and C are derived from g_0 , κ , s_0 , r_0 , and v (see [13]). By definition, an initial data set always satisfies the constraints and compatibility conditions. While the construction of initial data for the Einstein-Navier-Stokes equations is an important task, here our primary interest is on the evolution problem, and as such we shall take the standard approach of assuming the initial data as given (see the discussion in section 2.1).

We are now ready to state the main result. We refer the reader to the standard literature in General Relativity for the terminology employed in Theorem 1.4. We remind the reader of the definition of Gevrey spaces $\gamma^{m,(\sigma)}$ in Section 4, referring to references [50, 49, 72] for more details.

Theorem 1.4. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, v, s_0, r_0)$ be an initial data set for the incompressible Einstein-Navier-Stokes system (1.15), with Σ compact, $s_0 > 0$, $r_0 > 0$, and an equation of state \mathcal{P} such that (1.12) and (1.13) are satisfied. Let \mathcal{F}_∂ be a given smooth function of Ω , g and its derivatives up to second order, C and u and their derivatives up to first order, and assume that $q > 0$. Assume that the initial data is in $\gamma^{(\sigma)}(\Sigma)$ for some $1 \leq \sigma < \frac{24}{23}$. Then there exist a space-time (M, g) that is a development of \mathcal{I} , real valued functions $s > 0$ and $r > 0$ defined on M , and a vector field u , such that $g \in \gamma^{3,(\sigma)}(M)$, $u \in \gamma^{2,(\sigma)}(M)$, $s \in \gamma^{2,(\sigma)}(M)$, $r \in \gamma^{2,(\sigma)}(M)$, and (g, u, s, r) satisfy the incompressible Einstein-Navier-Stokes system in M .*

Furthermore, this solution satisfies the geometric uniqueness and domain of dependence properties, in the following sense. Let $\mathcal{I}' = (\Sigma', g'_0, \kappa', v', s'_0, r'_0)$ be another initial data set, also with equation of state \mathcal{P} , with corresponding development (M', g') and solution (g', u', s', r') of the incompressible Einstein-Navier-Stokes equations in M' . Assume that there exists a diffeomorphism between $S \subset \Sigma$ and $S' \subset \Sigma'$ that carries $\mathcal{I}|_S$ onto $\mathcal{I}'|_{S'}$, where S and S' are, respectively, domains in Σ and Σ' . Then there exists a diffeomorphism between $D_g(S) \subset M$ and $D_{g'}(S') \subset M'$ carrying (g, u, s, r) onto (g', u', s', r') , where $D_g(S)$ denotes the future domain of dependence of S in the metric g ; in particular $D_g(S)$ and $D_{g'}(S')$ are isometric.

Remark 1.5. Under the further assumption that the fluid is irrotational, Theorem (1.4) was proved by the second author in [13], where a better regularity result than in theorem 1.4, namely, $\sigma < 2$, was obtained.

Remark 1.6. The space-time M is diffeomorphic to $\Sigma \times [0, T]$ for some $T > 0$, and to $\Sigma \times [0, \tilde{T})$ for some $\tilde{T} > T$ if we require it to be a maximal Cauchy development.

Remark 1.7. The compactness of Σ is not absolutely necessary due to the domain of dependence property. However, in the case of a non-compact Σ without asymptotic conditions on the initial data, M may not contain any Cauchy surface other than Σ itself.

Remark 1.8. The hypotheses $s_0 > 0$ and $r_0 > 0$ guarantee, by continuity, the positivity of s and r in the neighborhood of Σ , as stated in the theorem. The assumption $s_0 > 0$ could be weakened to

$s_0 \geq 0$, but in this case the non-negativity of s in M would have to be derived from the equations of motion, a task we avoid for brevity. On the other hand, allowing r_0 to vanish would cause severe difficulties. In fact, the well-posedness of the corresponding problem is largely open even in the case of an ideal fluid [33].

In the following, we adopt:

Convention 1.9. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

2. DISCUSSION ON THE HYPOTHESES AND THE THESIS OF THEOREM 1.4

In this section we comment on the relevance of Theorem 1.4 for the study of relativistic fluids with viscosity. We highlight the restrictions imposed by (1.14) and the regularity of solutions, make some remarks regarding the physical content of the Theorem, and discuss how this work fits within the broader context of relativistic viscous fluids, making some general remarks about (1.1) and its particular case (1.8) along the way. Readers interested solely in the proof of Theorem 1.4 may skip this section.

2.1. The evolution of the vorticity. Let us start with the evolution condition imposed on Ω , i.e., equation (1.14) or, equivalently, (1.15e). In its full-generality, the incompressible Einstein-Navier-Stokes system consists of equations (1.15a)-(1.15d) and (1.15f), i.e., (1.15) without (1.15e). As such a system is rather complicated, we have imposed (1.14) which, of course, is ultimately a restriction on the unknowns (g, u, s, r) . From the point of view of (1.15a)-(1.15d), equation (1.15e) should be understood as a constraint, in the following sense. An initial data set yielding a solution to (1.15a)-(1.15d) (plus (1.15f)) will also give a solution to (1.15) only if further relations among the initial data hold. Indeed, arguing as in [13], one determines, from the original Cauchy data and Einstein's equations, the values of $\partial^2 g$, ∂s , ∂u , Ω , and $\partial^2 C$ (as well as the corresponding lower order derivatives) on the initial hypersurface $\{t = 0\}$, obtaining a relation of the form $\partial_0 \Omega_{\alpha\beta} = W_{\alpha\beta}$ on $\{t = 0\}$, where $W_{\alpha\beta}$ is a function of the Cauchy data. From (1.14), one also obtains a relation $\partial_0 \Omega_{\alpha\beta} = Z_{\alpha\beta}$ on $\{t = 0\}$, with $Z_{\alpha\beta}$ a function of the Cauchy data. Therefore the initial data must be such data $W_{\alpha\beta} = Z_{\alpha\beta}$, and hence Theorem 1.4 is ultimately a result under restrictions on the initial data, namely, the initial data ought to satisfy compatibility conditions imposed by (1.15e), as mentioned earlier⁷.

How large is the class of initial conditions satisfying the above restrictions is by no means an unimportant question, but one that is, at this point, premature, since we do not even know whether the system (1.15a)-(1.15d) and (1.15f) has any solution at all outside the class of analytic functions. While this leaves open the question of how general Theorem 1.4 is, it is consistent with some expected physical applications as discussed in section 2.4. Such restrictions notwithstanding, we make two important remarks.

First, one could, in principle, consider the case of zero vorticity, with the function \mathcal{F}_ϑ chosen so that (1.14) becomes an identity (notice that our results do not rely on any specific form of \mathcal{F}_ϑ , except for the dependence on the number of derivatives of its arguments). In this case, the constraints reduce to those that have to be imposed on the initial data when $\Omega = 0$. One could say then that our theorem reproves the earlier result [13]. While obviously this is not our goal here, it at least shows the set of appropriate initial data to be non-empty. The interesting question is whether the

⁷We notice that a similar, albeit notably simpler, situation happens to perfect fluids with zero vorticity. In that case, the equation for the vorticity is given by $\mathcal{L}_C \Omega = 0$, where \mathcal{L} is the Lie derivative. This is not compatible with the equations derived from the divergence of the stress-energy tensor unless the initial data is such that $\Omega = 0$ on the $\{t = 0\}$ slice.

set of initial data satisfying the constraints and compatibility conditions is non-empty modulo zero vorticity. This will be addressed in a future work.

Second, in light of how little is known about viscosity in General Relativity (see section 2.3), our conditional result should be viewed as evidence that further investigation of Lichnerowicz’s proposal (1.1) is a worthwhile line of inquiry. In this regard, it is illustrative to point out that there are other situations in General Relativity where the evolution problem is investigated without a decisive answer to the question of solvability of the constraints, but this has never stopped the community to make conditional statements regarding Einstein’s equations. One such situation, for example, is the study of vacuum Einstein’s equations with low regularity. As discussed, for instance, in [58], the “rough solution theory of the constraints has in fact lagged behind that of the evolution problem.” For instance, well-posedness of the vacuum evolution problem for data (g, κ) in $H^s \times H^{s-1}$, $s > \frac{5}{2}$, had been known since 1977 [42]. However, it was not until 2004 that solutions to the constraint equations in this regularity class could be constructed [7]. Hence, for 27 years it was not known if the classical result [42] was not empty modulo large values of s (for which [42] would simply reproduce earlier known results).

2.2. Regularity of solutions. We work in the Gevrey class, because the equations we derive form a Leray-Ohya system (see [49]), which, in general, is not well-posed in Sobolev spaces. Gevrey spaces have become an important tool in analyzing the equations of Fluid Dynamics, especially when viscosity is present (see, e.g., [3, 4, 25, 27, 72] and references therein). Hence, it is sensible that such spaces might play a role in the case of relativistic viscous fluids as well. Furthermore, Gevrey spaces are not completely foreign to the study of Einstein’s equations: in some relevant circumstances, the equations of ideal magneto-hydrodynamics appear to have been shown to be well-posed only in the Gevrey class [8, 33]⁸. On the other hand, the overwhelming success of Sobolev space techniques in the investigation of the Cauchy problem for Einstein’s equations⁹ almost demands that we employ Sobolev spaces in the study of the evolution problem. Moreover, in order to eventually settle the question of whether (1.1) can give a physically satisfactory description of relativistic viscous phenomena, we have to be able to explicitly compute several physical observables. For this, one has to solve the equations numerically, which, in turn, requires that the equations be well-posed in some function space characterized by a finite number of derivatives.

Unfortunately, currently Gevrey regularity seems to be the best one can do for (1.1), as the corresponding equations of motion do not seem to be amenable to known Sobolev-type techniques. We remark, however, that simply establishing causality of the equations of motion is already a step forward in light of the long history of non-causal theories of relativistic viscous fluids [70].

2.3. The status of viscosity in relativity and Lichnerowicz’s proposal. In spite of the restriction on the initial data due to (1.14), the severity of which we acknowledge is yet to be understood, and on the regularity class of solutions, one should not overlook the conclusion of Theorem 1.4: it is possible, employing the traditional Standard Irreversible Thermodynamics, to obtain a description of relativistic viscous fluids that is well-posed and does not exhibit faster than light signals. In this regard, we remind the reader once more that we are attempting a new look at this problem through a first order formalism. Hence, it is all but unreasonable to start off with conditions that render the problem tractable with current mathematical technology. The first attempt in this direction [13] dealt with irrotational fluids. Here, we considered a less dramatic condition on the vorticity, namely, (1.14), which seems to be compatible with some physical applications (see section 2.4). The message

⁸Although it is very likely that the formulation of [2] would carry over, with almost no modifications, to the coupling with Einstein’s equations. A proof of this statement, however, does not seem to be available in the literature.

⁹The literature on this topic is too vast; see, e.g., the monographs [8, 71].

conveyed by this is that, while it is wide open whether a full existence result for the incompressible Einstein-Navier-Stokes may be within reach without restrictions on the vorticity, one can still prove well-posedness results under interesting scenarios.

Another restriction in our Theorem that one would like to remove is the incompressibility hypothesis, not only for the sake of mathematical generality, but also because relativistic systems many times exhibit sound waves that propagate at sub-luminal speeds. This is the subject of ongoing investigations.

In order to put all of the above in perspective, we give rather brief overview of what is currently known about viscosity in relativity.

The Mueller-Israel-Stewart (MIS) theory [43, 44, 45, 46, 62, 75] is probably the best accepted theory of relativistic viscous phenomena. It consists of a systematic application of the ideas of Relativistic Extended Irreversible Thermodynamics [47, 63]. The linearization about equilibrium states of the MIS theory has been shown to be causal [39]. The non-linear theory, however, is also plagued with non-causality behavior [41]. To be fair, such loss of causality is known to happen under extreme physical conditions unlikely to be met by most realistic systems. More precisely, in [41] the authors investigate the relatively simple case where only heat conduction is present, so that the bulk and shear viscosity are zero, and under the assumption of planar symmetry. Under these assumptions, it is shown in [41] that the equations of motion are causal under a restriction on the values of the heat-flux, and non-causal if such a restriction is violated¹⁰. In contrast, Theorem 1.4, as well as [13], makes no symmetry or near-equilibrium assumption, and treats the full non-linear system, albeit it assumes stiffness and stringent restrictions on the initial data (or irrotationality in the case of [13]). It is important to notice, however, that from the point of view of causality, such results treat precisely the most “dangerous” scenario, i.e., they include the term

$$\pi_\alpha^\rho \pi_\beta^\mu (\nabla_\rho C_\mu + \nabla_\mu C_\rho),$$

which leads to multiple characteristic due to the presence of the projections $\pi_\alpha^\rho \pi_\beta^\mu$. The causality obtained in [41], on the other hand, is restricted to the case when the viscous part of the stress-energy tensor contains only the heat flow; in particular, such projection terms are absent.

We also point out that, to the best of our knowledge, the aforementioned causality and well-posedness results of the MIS theory [39, 41] do not include coupling to Einstein’s equations, i.e., they consider the fluid equations in a fixed background (except for some very simple situations such as FRW cosmologies [59]), whereas Theorem 1.4 and [13] do treat the full Einstein-fluid system.

Another interesting feature of Theorem 1.4 is that it circumvents the instability results of Hiscock and Lindblom [40]. In fact, formally the equations that we study here correspond to the case $\kappa = \sigma = 0$ in [40]. Equations (1.7) and (1.14), however, further constrain the evolution of perturbations (compare with equations (41) in [40]). On the other hand, if condition (1.7) is dropped, then the term $\nabla_\mu C^\mu$ that contributes to the viscous part in (1.1) will depend on derivatives of the thermodynamic variables along the flow, a case not covered under the assumptions of [40].

One important question about theories based on (1.1) is whether natural physical requirements are satisfied. One such requirement is that entropy production be non-negative. It is not difficult to see that, at least for the case investigated here, namely, when (1.1) reduces to (1.8), this is the case. To see this, one first uses $u^\beta \nabla^\alpha T_{\alpha\beta} = 0$ and the first law of thermodynamics to derive

$$\theta r u^\alpha \partial_\alpha s = \vartheta F (\nabla^\mu u^\nu \nabla_\mu u_\nu + \nabla^\mu u^\nu \nabla_\nu u_\mu - u^\mu \nabla_\mu u^\alpha u^\nu \nabla_\nu u_\alpha).$$

¹⁰In passing, one should note that the MIS is sometimes referred to as “causal dissipative relativistic theory,” but strictly speaking that is, in view of what has been said, a misnomer.

On the other hand, direct computation gives

$$\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} = 2F^2(\nabla^\mu u^\nu \nabla_\mu u_\nu + \nabla^\mu u^\nu \nabla_\nu u_\mu - u^\mu \nabla_\mu u^\alpha u^\nu \nabla_\nu u_\alpha),$$

thus

$$\theta r u^\alpha \partial_\alpha s = \frac{\vartheta}{2F} \Sigma^{\alpha\beta} \Sigma_{\alpha\beta} \geq 0,$$

since $\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} \leq 0$ ¹¹ and $\vartheta \leq 0$. A detailed study of physical consistency of models based on (1.1) appears in [16] (see [15], however, for some further results concerning physical properties of (1.1)).

In summary, it is fair to say that despite considerable progress since the original work of Eckart [23], the description of relativistic viscous phenomena still presents many challenges. These remarks are not intended to claim that Lichnerowicz's proposal is better than the more studied MIS theory or should be favored over any other theory, but rather to highlight how little is known about viscosity in General Relativity, which makes, in our opinion, attempts at different approaches, such as those based on (1.1), welcome.

2.4. Vorticity in relativistic fluids and some physical considerations. In the case of inviscid fluids, the vorticity obeys [53, 55]

$$\mathcal{L}_C \Omega_{\alpha\beta} + \partial_\alpha(\theta F) \partial_\beta s - \partial_\beta(\theta F) \partial_\alpha s = 0. \quad (2.1)$$

This equation, sometimes called the Lichnerowicz, or Carter-Lichnerowicz equation, plays an important role in the study of inviscid relativistic fluids, and generalizations have also been employed in formulations with viscosity [1, 11, 18, 19, 35, 38, 51, 74, 77].

Equation (1.14) reduces to (2.1) when $\mathcal{F}_\vartheta = 0 = q$; thus, in particular, we see that in physically relevant models, q and \mathcal{F}_ϑ vanish when viscosity is absent. Many of the modifications of (2.1) that include viscosity tend to occur in the context of very specific models, where the equations are simple as compared to the ones here investigated (for instance, perturbations of an FRW model). These can generally be accommodated by (1.14) with a suitable choice of \mathcal{F}_ϑ ; see, for example, [1, 11, 18, 19, 37, 38, 74] and references therein. In more general terms, (1.14) seems natural when one considers applications with small viscosity, in that the evolution of Ω should, to a certain degree, resemble that of an ideal fluid. We also point out that (1.14) is consistent with standard cosmology (with no viscosity), in that, in such scenarios, Ω decays with the Hubble expansion, being, as a consequence, ignored in many circumstances¹². Hence, one may, again, suspect that in these cases Ω will be governed by an equation similar to that of ideal fluids, since (2.1) enjoys the property of preserving zero vorticity. Our choice (1.14) is a compromise between the previous considerations and an algebraic form for which properties of hyperbolic polynomials, necessary for our techniques, hold true. See section 4.

3. A NEW SYSTEM OF EQUATIONS.

Here we derive a different system of equations, whose existence of solutions implies Theorem 1.4. Thus, for the rest of this section, we assume we have a sufficiently differentiable solution to (1.15). In particular, in light of (1.2) and (1.15f), one has

$$F = \sqrt{C^\mu C_\mu},$$

so that F can be viewed as a function of g and C .

¹¹Recall our convention for the metric.

¹²Although vorticity may play an important role in early stages of the Universe. See, for example, [11] and references therein.

Convention 3.1. Unless stated otherwise, we shall assume from now on to be working in harmonic (or wave) coordinates.

Notation 3.2. Below, B , with indices attached when necessary, is used to denote expressions, where the maximum number of the derivatives of the variables g , s , u , Ω , F , and C is indicated in the arguments. For instance, $B(\partial g, \partial^2 s)$ indicates an expression depending on at most first derivatives of g and second derivatives of s . The expression represented by B can vary from equation to equation.

3.1. Equation for g . By taking the trace of (1.15a) we get

$$T := g^{\alpha\beta} T_{\alpha\beta} = -R + 4\Lambda,$$

so we can rewrite Einstein's equation as

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} + \Lambda g_{\alpha\beta}. \quad (3.1)$$

Next, recall that in harmonic coordinates, the Ricci curvature can be written as

$$R_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \partial_{\mu\nu} g_{\alpha\beta} + B_{\alpha\beta}(\partial g). \quad (3.2)$$

From (1.9) we also have

$$\begin{aligned} T_{\alpha\beta} &= r F u_\alpha u_\beta - p g_{\alpha\beta} + F \vartheta \pi_\alpha^\rho \pi_\beta^\mu (\nabla_\rho u_\mu + \nabla_\mu u_\rho) \\ &= B_{\alpha\beta}(\partial g, s, u, C) + \vartheta \sqrt{C^\nu C_\nu} \pi_\alpha^\rho \pi_\beta^\mu (\partial_\rho u_\mu + \partial_\mu u_\rho). \end{aligned} \quad (3.3)$$

Hence

$$T = r F - 4p + \vartheta F \pi^{\rho\mu} (\nabla_\rho u_\mu + \nabla_\mu u_\rho) = \vartheta \sqrt{C^\nu C_\nu} \pi^{\rho\mu} (\partial_\rho u_\mu + \partial_\mu u_\rho) + B(\partial g, s, u, C). \quad (3.4)$$

Inserting (3.2), (3.3), and (3.4) into (3.1) we obtain the following equation for g

$$g^{\mu\nu} \partial_{\mu\nu} g_{\alpha\beta} - \vartheta \sqrt{C^\nu C_\nu} \left(2\pi_\alpha^\rho \pi_\beta^\mu (\partial_\rho u_\mu + \partial_\mu u_\rho) - \pi^{\rho\mu} (\partial_\rho u_\mu + \partial_\mu u_\rho) g_{\alpha\beta} \right) + B_{\alpha\beta}(\partial g, s, u, C) = 0. \quad (3.5)$$

3.2. Equation for s . From (1.15b), (1.15c), (1.6) by considering $u^\beta \nabla^\alpha T_{\alpha\beta}$ with $T_{\alpha\beta}$ as in (1.9), we obtain

$$r \theta u^\alpha \partial_\alpha s - \vartheta F (\nabla_\mu u_\nu + \nabla_\nu u_\mu) \pi^{\alpha\mu} \nabla_\alpha u^\nu = 0,$$

where we also used (1.16) and that $u^\beta \pi_\beta^\mu = 0$.

To obtain the desired quasi-linear structure we apply $u^\sigma \nabla_\sigma$ to the equation. This results in

$$\begin{aligned} u^\sigma u^\alpha \partial_{\alpha\sigma} s - \vartheta \frac{\sqrt{C^\rho C_\rho}}{\theta r} \pi^{\alpha\mu} u^\sigma \partial_\alpha u^\nu (\partial_{\mu\sigma} u_\nu + \partial_{\sigma\nu} u_\mu) - \vartheta \frac{\sqrt{C^\rho C_\rho}}{\theta r} (\partial_\mu u_\nu + \partial_\nu u_\mu) \pi^{\alpha\mu} u^\sigma \partial_{\alpha\sigma} u^\nu \\ + B(\partial^2 g, \partial s, \partial C, \partial u) = 0. \end{aligned} \quad (3.6)$$

We note that in the derivation of B in (3.6) at first one obtains derivatives in θ and r , which get replaced by ∂s , and ∂F . Then in view of the comment at the beginning of this section, ∂F gets replaced by ∂C and ∂g .

3.3. Equation for u . The derivation of the equation for u is long, requiring several calculations. We shall break them into short claims in order to facilitate the reading.

Since $F > 0$, inspired by [52] we can define the conformal metric

$$\bar{g} = F^2 g,$$

and denote by $\bar{\nabla}$ covariant differentiation in the \bar{g} -metric. We also let

$$\bar{C}^\alpha = F^{-1} u^\alpha, \quad (3.7)$$

i.e., \bar{C}^α is C_α with index raised in the \bar{g} metric, so that

$$\bar{C}^\alpha C_\alpha = 1. \quad (3.8)$$

It also follows that

$$\Omega_{\alpha\beta} = \bar{\nabla}_\alpha C_\beta - \bar{\nabla}_\beta C_\alpha. \quad (3.9)$$

If v is a one-form, a direct calculation gives

$$\bar{\nabla}_\alpha v_\beta = \nabla_\alpha v_\beta - K_\alpha v_\beta - K_\beta v_\alpha + K^\rho v_\rho g_{\alpha\beta}, \quad (3.10)$$

where $K_\alpha = \partial_\alpha \log F = \frac{\partial_\alpha F}{F}$. The following standard identities will also be needed,

$$\nabla_\alpha \nabla_\beta v^\lambda - \nabla_\beta \nabla_\alpha v^\lambda = R_{\alpha\beta}{}^\lambda{}_\gamma v^\gamma,$$

from which it follows

$$\nabla_\alpha \nabla_\beta v^\alpha - \nabla_\beta \nabla_\alpha v^\alpha = R_{\beta\gamma} v^\gamma. \quad (3.11)$$

To derive the equation for u we need to compute the divergence of $T_{\alpha\beta}$. For this it will be convenient to set

$$\Sigma_{\alpha\beta} = \pi_\alpha^\mu \pi_\beta^\nu (\nabla_\mu C_\nu + \nabla_\nu C_\mu), \quad (3.12)$$

which can be written as

$$\Sigma_{\alpha\beta} = F \pi_\alpha^\mu \pi_\beta^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu), \quad (3.13)$$

since $u^\beta \pi_\beta^\mu = 0$. $\Sigma_{\alpha\beta}$ is sometimes called the shear tensor.

Claim 3.3. *Let*

$$\bar{\Sigma}_{\alpha\beta} = \bar{\nabla}_\alpha C_\beta + \bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\bar{\nabla}_\lambda C_\alpha C_\beta + \bar{\nabla}_\lambda C_\beta C_\alpha).$$

Then the following two identities hold:

$$\bar{\Sigma}_{\alpha\beta} = \Sigma_{\alpha\beta} + 2\pi_{\alpha\beta} u^\rho \partial_\rho F,$$

and

$$\bar{\Sigma}_{\alpha\beta} = 2\bar{\nabla}_\beta C_\alpha + \Theta_{\alpha\beta},$$

where

$$\Theta_{\alpha\beta} = \Omega_{\alpha\beta} - u^\lambda (\Omega_{\lambda\alpha} u_\beta + \Omega_{\lambda\beta} u_\alpha).$$

Proof. Using (3.10), (1.2), (3.7), a long but not difficult calculation gives

$$\bar{\Sigma}_{\alpha\beta} = F(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - u_\beta u^\nu \nabla_\nu u_\alpha - u_\alpha u^\mu \nabla_\mu u_\beta) + 2\pi_{\alpha\beta} u^\rho \partial_\rho F.$$

But in light of (1.16) we have

$$\pi_\alpha^\mu \pi_\beta^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu) = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - u_\beta u^\nu \nabla_\nu u_\alpha - u_\alpha u^\mu \nabla_\mu u_\beta,$$

so that (3.13) gives

$$\bar{\Sigma}_{\alpha\beta} = \Sigma_{\alpha\beta} + 2\pi_{\alpha\beta} u^\rho \partial_\rho F.$$

For the second identity, use (3.9) to get

$$\begin{aligned} \bar{\Sigma}_{\alpha\beta} &= \bar{\nabla}_\alpha C_\beta + \bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\bar{\nabla}_\lambda C_\alpha C_\beta + \bar{\nabla}_\lambda C_\beta C_\alpha) \\ &= \Omega_{\alpha\beta} + 2\bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\Omega_{\lambda\alpha} C_\beta + \Omega_{\lambda\beta} C_\alpha) - \bar{C}^\lambda (\bar{\nabla}_\alpha C_\lambda C_\beta + \bar{\nabla}_\beta C_\lambda C_\alpha). \end{aligned}$$

The result now follows by noticing that (3.8) gives $\bar{C}^\lambda \bar{\nabla}_\alpha C_\lambda = 0 = \bar{C}^\lambda \bar{\nabla}_\beta C_\lambda$ and using (1.2). \square

Claim 3.4. *We have*

$$\pi^{\gamma\beta} \nabla_\alpha \bar{\Sigma}^\alpha_\beta = -2\pi^{\gamma\beta} K^\alpha \Omega_{\alpha\beta} + \pi^{\gamma\beta} \nabla_\alpha \Theta^\alpha_\beta.$$

Proof. Using claim 3.3, (3.10), (3.11), and (1.15d), one gets

$$\begin{aligned} \nabla_\alpha \bar{\Sigma}^\alpha_\beta &= 2\nabla_\alpha \nabla_\beta C^\alpha - 2\nabla_\alpha K^\alpha C_\beta - 2K^\alpha \Omega_{\alpha\beta} + \nabla_\alpha \Theta^\alpha_\beta \\ &= 2R_{\alpha\beta} C^\alpha - 2\nabla_\alpha K^\alpha C_\beta - 2K^\alpha \Omega_{\alpha\beta} + \nabla_\alpha \Theta^\alpha_\beta. \end{aligned}$$

Now, from (3.1), the fact that $\pi_{\alpha\beta} C^\alpha = F\pi_{\alpha\beta} u^\alpha = 0$, and the form of $T_{\alpha\beta}$, it follows that

$$R_{\alpha\beta} C^\alpha = (rF - p - \frac{1}{2}T)C_\beta,$$

so that

$$\nabla_\alpha \bar{\Sigma}^\alpha_\beta = (2rF - 2p - T - 2\nabla_\alpha K^\alpha)C_\beta - 2K^\alpha \Omega_{\alpha\beta} + \nabla_\alpha \Theta^\alpha_\beta,$$

from which the claim follows upon contracting with $\pi^{\gamma\beta}$ and using again $\pi_{\alpha\beta} C^\alpha = 0$ \square

Claim 3.5. *We have*

$$2u^\alpha \pi^{\gamma\rho} \nabla_\alpha \partial_\rho F = -2\pi^{\gamma\beta} K^\alpha \Omega_{\alpha\beta} + \pi^{\gamma\beta} \nabla_\alpha \Theta^\alpha_\beta + B^\gamma (\partial g, \partial s, \partial F, \partial u).$$

Proof. Combining the first identity of claim 3.3 with claim 3.4,

$$\pi^{\gamma\beta} \nabla_\alpha \Sigma^\alpha_\beta = -2u^\rho \pi^{\gamma\alpha} \nabla_\alpha \partial_\rho F - 2\pi^{\gamma\beta} K^\alpha \Omega_{\alpha\beta} + \pi^{\gamma\beta} \nabla_\alpha \Theta^\alpha_\beta + B^\gamma (\partial g, \partial F, \partial u).$$

Writing (1.8) as $T_{\alpha\beta} = \hat{T}_{\alpha\beta} + \vartheta \Sigma_{\alpha\beta}$, noticing that $\pi^{\gamma\beta} \nabla_\alpha \hat{T}_{\alpha\beta} = B^\gamma (\partial g, \partial F, \partial u)$, and invoking (1.15b), we have

$$\pi^{\gamma\beta} \nabla_\alpha \Sigma^\alpha_\beta = B^\gamma (\partial g, \partial s, \partial F, \partial u), \quad (3.14)$$

since $\vartheta > 0$. The claim follows from these last two equalities. \square

Claim 3.6. *We have*

$$u^\alpha \nabla_\alpha u_\beta = \pi_\beta^\alpha \frac{\partial_\alpha F}{F} + \frac{1}{F} u^\alpha \Omega_{\alpha\beta}.$$

Proof. From (3.10) and (1.2),

$$\bar{\nabla}_\alpha C_\beta = F \nabla_\alpha u_\beta - \partial_\beta F u_\alpha + u^\rho \partial_\rho F g_{\alpha\beta},$$

which upon contraction gives

$$\bar{C}^\alpha \bar{\nabla}_\alpha C_\beta = u^\alpha \nabla_\alpha u_\beta - \pi_\beta^\alpha \frac{\partial_\alpha F}{F}.$$

Contracting (3.9) with \bar{C}^α , using (3.8) and the above equality, one obtains the result, after rewriting C in terms of F and u . \square

Claim 3.7. *We have*

$$\begin{aligned} 2\pi^{\gamma\beta} \nabla^\alpha \Sigma_{\alpha\beta} &= 2F \nabla^\alpha \nabla_\alpha u^\gamma - \pi^{\gamma\beta} \nabla_\alpha \Theta^\alpha{}_\beta - 2u^\alpha u^\rho \nabla_\alpha \Omega_\rho{}^\gamma \\ &\quad + 2\pi^{\alpha\mu} \pi^{\gamma\nu} \nabla_\alpha \nabla_\nu C_\mu + B^\gamma (\partial^2 g, \partial s, \partial F, \partial u, \Omega, \partial C). \end{aligned}$$

Proof. Use (1.2) in the first term on the right hand side of (3.12) to write it as

$$\Sigma_{\alpha\beta} = F \pi_\alpha^\mu \pi_\beta^\nu \nabla_\mu u_\nu + \pi_\alpha^\mu \pi_\beta^\nu \nabla_\nu C_\mu,$$

where $\pi_\beta^\nu u_\nu = 0$ has been employed. Applying $\pi^{\gamma\beta} \nabla^\alpha$, we get

$$\pi^{\gamma\beta} \nabla^\alpha \Sigma_{\alpha\beta} = F (\nabla_\alpha \nabla^\alpha u^\gamma - u^\alpha u^\mu \nabla_\alpha \nabla_\mu u^\gamma) + \pi^{\alpha\mu} \pi^{\gamma\nu} \nabla_\alpha \nabla_\nu C_\mu + B^\gamma (\partial g, \partial F, \partial u, \partial C), \quad (3.15)$$

where we have used that (1.16) implies

$$u^\nu \nabla^\alpha \nabla_\alpha u_\nu = \nabla^\alpha (u^\nu \nabla_\alpha u_\nu) - \nabla^\alpha u^\nu \nabla_\alpha u_\nu = B^\gamma (\partial g, \partial u),$$

and

$$u^\nu \nabla_\alpha \nabla_\mu u_\nu = \nabla_\alpha (u^\nu \nabla_\mu u_\nu) - \nabla_\alpha u^\nu \nabla_\mu u_\nu = B^\gamma (\partial g, \partial u).$$

Commuting u^μ and ∇_α one obtains, in light of claim 3.6,

$$F u^\alpha u^\mu \nabla_\alpha \nabla_\mu u^\gamma = u^\alpha \pi^{\gamma\rho} \nabla_\alpha \partial_\rho F + u^\rho u^\alpha \nabla_\alpha \Omega_\rho{}^\gamma + B^\gamma (\partial g, \partial F, \partial u, \Omega),$$

so that (3.15) becomes

$$\begin{aligned} \pi^{\gamma\beta} \nabla^\alpha \Sigma_{\alpha\beta} &= F \nabla_\alpha \nabla^\alpha u^\gamma - u^\alpha \pi^{\gamma\rho} \nabla_\alpha \partial_\rho F - u^\rho u^\alpha \nabla_\alpha \Omega_\rho{}^\gamma \\ &\quad + \pi^{\alpha\mu} \pi^{\gamma\nu} \nabla_\alpha \nabla_\nu C_\mu + B^\gamma (\partial g, \partial F, \partial u, \partial C, \Omega). \end{aligned}$$

The result now follows by using claim 3.5 to eliminate $u^\alpha \pi^{\gamma\rho} \nabla_\alpha \partial_\rho F$ from the above expression, after noticing that $K^\alpha \Omega_{\alpha\beta}$ can be absorbed into B^γ . \square

In light of (3.14) and using the definition of Θ , claim 3.7 gives the desired equation for u , namely,

$$\begin{aligned} g^{\mu\nu} \partial_{\mu\nu} u_\gamma - \frac{1}{2\sqrt{C^\rho C_\rho}} g^{\mu\nu} \partial_\nu \Omega_{\mu\gamma} - \frac{1}{2\sqrt{C^\rho C_\rho}} u^\mu u^\nu \partial_\mu \Omega_{\nu\gamma} + \frac{1}{2\sqrt{C^\rho C_\rho}} u_\gamma u^\mu g^{\nu\beta} \partial_\beta \Omega_{\nu\mu} \\ + \frac{1}{\sqrt{C^\rho C_\rho}} \pi^{\alpha\mu} \pi_\gamma^\nu \partial_{\alpha\nu} C_\mu + B_\gamma (\partial^2 g, \partial s, \partial u, \Omega, \partial C) = 0, \end{aligned} \quad (3.16)$$

where we used $F > 0$, and subsequently (1.2) to eliminate the F dependence.

3.4. Equations for Ω . Recalling that

$$\mathcal{L}_C \Omega_{\alpha\beta} = C^\mu \nabla_\mu \Omega_{\alpha\beta} + \nabla_\alpha C^\mu \Omega_{\mu\beta} + \nabla_\beta C^\mu \Omega_{\alpha\mu},$$

we see that (1.14) has the form

$$C^\mu \partial_\mu \Omega_{\alpha\beta} + q u^\mu \partial_{\mu\alpha} C_\beta - q u^\mu \partial_{\mu\beta} C_\alpha + B_{\alpha\beta} (\partial^2 g, \partial s, \partial u, \Omega, \partial C) = 0. \quad (3.17)$$

3.5. Equations for C . In order to close the system, we need to specify equations of motion for C . Since all the equations from (1.15) have already been employed above in the derivation of equations for g , s , u , and Ω , one suspects that equations for C should be determined by some extra conditions, not explicitly present in (1.15). However, in order to do so without changing the content of the original problem, one must choose equations that are necessarily satisfied by any solution to (1.15). Thus, convenient identities, that follow from standard tensor calculus and our basic definitions, will be employed.

Using the definition of the Hodge-Laplacian gives

$$\Delta C = d\delta C + \delta dC = \delta\Omega,$$

where $\delta C = -\nabla^\mu C_\mu = 0$ (see (1.15d)) and the definition of Ω , (1.10), have been used. On the other hand, recalling

$$(\Delta C)_\alpha = -\nabla^\mu \nabla_\mu C_\alpha + R_{\mu\alpha} C^\mu,$$

we obtain the following equation for C :

$$g^{\mu\nu} \partial_{\mu\nu} C_\alpha - g^{\mu\nu} \partial_\mu \Omega_{\nu\alpha} + B_\alpha(\partial^2 g, \Omega, \partial C) = 0. \quad (3.18)$$

3.6. The full system. The sought new system of equations consists of (3.5), (3.6), (3.16), (3.17), and (3.18). These are 25 equations for the 25 unknowns: ten $g_{\alpha\beta}$, one s , four u_α , six $\Omega_{\alpha\beta}$, and four C_α . We shall refer to this system as the modified incompressible Einstein-Navier-Stokes system. One important aspect of our proof consists in showing that the modified incompressible Einstein-Navier-Stokes system forms a Leray-Ohya system [49], which depends, among other things, on a counting of derivatives. For this purpose, it is convenient to write the system symbolically as

$$\left\{ \begin{array}{llll} a_{11}(g)\partial^2 g & + & 0 & + & a_{13}(g, u, C)\partial u & + & 0 \\ 0 & + & a_{22}(g, u)\partial^2 s & + & a_{23}(g, s, \partial u, C)\partial^2 u & + & 0 \\ 0 & + & 0 & + & a_{33}(g)\partial^2 u & + & a_{34}(g, u, C)\partial\Omega \\ 0 & + & 0 & + & 0 & + & a_{44}(g, C)\partial\Omega \\ 0 & + & 0 & + & 0 & + & a_{54}(g)\partial\Omega \end{array} \right.$$

$$+ \quad 0 \quad = B_g(\partial g, s, u, C), \quad (3.19a)$$

$$+ \quad 0 \quad = B_s(\partial^2 g, \partial s, \partial u, \partial C), \quad (3.19b)$$

$$+ a_{35}(g, u, C)\partial^2 C = B_u(\partial^2 g, \partial s, \partial u, \Omega, \partial C), \quad (3.19c)$$

$$+ a_{45}(g, u)\partial^2 C = B_\Omega(\partial^2 g, \partial s, \partial u, \Omega, \partial C), \quad (3.19d)$$

$$+ a_{55}(g)\partial^2 C = B_C(\partial^2 g, \Omega, \partial C). \quad (3.19e)$$

We write this more succinctly as

$$A(V, \partial)V = B(V),$$

where $V = (g, s, u, \Omega, C)$, $B(V) = (B_g, B_s, B_u, B_\Omega, B_C)$, and

$$A(V, \partial) = \begin{pmatrix} a_{11}(g)\partial^2 & 0 & a_{13}(g, u, C)\partial & 0 & 0 \\ 0 & a_{22}(g, u)\partial^2 & a_{23}(g, s, \partial u, C)\partial^2 & 0 & 0 \\ 0 & 0 & a_{33}(g)\partial^2 & a_{34}(g, u, C)\partial & a_{35}(g, u, C)\partial^2 \\ 0 & 0 & 0 & a_{44}(g, C)\partial & a_{45}(g, u)\partial^2 \\ 0 & 0 & 0 & a_{54}(g)\partial & a_{55}(g)\partial^2 \end{pmatrix} \quad (3.20)$$

4. PROOF OF THEOREM 1.4.

The main tool in the proof of theorem 1.4 is the theory of weakly hyperbolic equations in Gevrey spaces developed by Leray and Ohya [50, 49, 64] and extended by Choquet-Bruhat [6] to the type of non-diagonal systems which will be of interest here. We shall not review these constructions, except for those aspects that will be necessary to fix our notation and conventions, referring the reader to the above references for the complete discussion. Some aspects of the Leray-Ohya theory can also be found (without proofs) in [8, 9, 13].

For the reader's convenience, we start by recalling the definition of Gevrey spaces. As our proof is essentially local, with a global (in space) solution obtained by a gluing argument, it suffices to give the definition in the case of \mathbb{R}^{n+1} , whose coordinates we denote by (x_0, \dots, x_n) . For a number $|X| > 0$, let X be the strip $0 \leq x^0 \leq |X|$. The Gevrey space $\gamma^{m,(\sigma)}(X)$ is defined as follows. Let $S_t = \{x^0 = t\}$, and

$$|D^k u|_t = c(n, k) \sup_{|\alpha| \leq k} \|D^\alpha u\|_{L^2(S_t)},$$

where $c(n, k)$ is a normalization constant. Then, for $\sigma \geq 1$, and m a non-negative integer, $u \in \gamma^{m,(\sigma)}(X)$ means that $u \in C^\infty(X)$, and

$$\sup_{|\beta| \leq m, \alpha, 0 \leq t \leq |X|} \frac{1}{(1 + |\alpha|)^\sigma} \left(|D^{\beta+\alpha} u|_t \right)^{\frac{1}{1+|\alpha|}} < \infty,$$

where the sup over α is taken over multi-indices such that $\alpha_0 = 0$. Analogously one defines such spaces for open sets, product spaces, etc. Intuitively, $\gamma^{m,(\sigma)}(X)$ can be thought of as a space between analytic and smooth functions, in the following sense. An analytic function u on S_t obeys, on each compact set, an inequality of the form $|D^\alpha u| \leq C^{|\alpha|+1} \alpha!$, for some $C > 0$. Gevrey functions of class $\sigma \geq 1$ are smooth functions obeying the weaker inequality $|D^\alpha u| \leq C^{|\alpha|+1} (\alpha!)^\sigma$. Then, $\gamma^{m,(\sigma)}(X)$ consists of those functions whose derivatives up to order m , restricted to each time slice S_t , belong to the Gevrey space of class σ — except that it is convenient to characterize the Gevrey spaces of S_t with the help of an integral norm, as done above. Gevrey spaces of functions defined on a hypersurface $\Sigma \subset X$ (say, on $\{x^0 = 0\}$), are defined in an analogous fashion and denoted by $\gamma^{(\sigma)}(\Sigma)$. These will be the spaces where initial data is prescribed (notice that there is no supremum over β in this case). Gevrey spaces are important in particular because it is possible to establish in them the well-posedness of certain PDEs that are known not to be well-posed in Sobolev or smooth spaces [61]. At the same time, Gevrey spaces allow constructions with compactly supported functions, an important tool in analysis not possible in the class of analytic functions. See [49, 72] for details on Gevrey spaces and their applications.

Consider a system of N partial differential equations and N unknowns in $X = \mathbb{R}^n \times [0, T]$, and denote the unknown as $V = (v^I)$, $I = 1, \dots, N$. Suppose that the system has the following quasi-linear structure: it is possible to attach to each unknown v^I a non-negative integer m_I , and to each equation a non-negative integer n_J , such that the system reads

$$h_I^J(\partial^{m_K - n_J - 1} v^K, \partial^{m_I - n_J} v^I) + b^J(\partial^{m_K - n_J - 1} v^K) = 0, \quad J = 1, \dots, N. \quad (4.1)$$

The notation here is similar to the one we used to write system (3.19), namely, $h_I^J(\partial^{m_K - n_J - 1} v^K, \partial^{m_I - n_J} v^I)$ is a homogeneous differential operator of order $m_I - n_J$ (which can be zero), whose coefficients depend on at most $m_K - n_J - 1$ derivatives of v^K , $K = 1, \dots, N$. The remaining terms, $b^J(\partial^{m_K - n_J - 1} v^K)$, also depend on at most $m_K - n_J - 1$ derivatives of v^K , $K = 1, \dots, N$.

Recall that the characteristic polynomial of (4.1) at $x \in X$ and for a given V is the polynomial in the co-tangent space T_x^*X , $p(\xi)$, $\xi \in T_x^*X$, of degree

$$\ell = \sum_{I=1}^N m_I - \sum_{J=1}^N n_J,$$

given by the principal part (of order ℓ) of the characteristic determinant of the system, $\det(h_I^J(\xi))$.

Consider the Cauchy problem for (4.1), with Cauchy data given on $X_0 = \mathbb{R}^n \times \{t = 0\}$. Assume that for any $x \in X_0$, and with V taking the values of the Cauchy data on X_0 , the characteristic polynomial $p(\xi)$ is a product of q hyperbolic polynomials of orders ℓ_q ,

$$p(\xi) = p_1(\xi) \cdots p_q(\xi).$$

Suppose finally that

$$\max_q \ell_q \geq \max_I m_I - \min_J n_J.$$

Building on the techniques developed in [49] Choquet-Bruhat proved [6] under the above conditions, the Cauchy problem for (4.1) has a unique solution V in the Gevrey space $\gamma^{m,(\sigma)}(X')$, where $X' = \mathbb{R}^n \times [0, T']$, $T' \leq T$, for a suitable integer $m \geq 0$, and $1 \leq \sigma < \sigma_0 = \frac{q}{q-1}$ (the case $q = 1$, $\sigma_0 = \infty$, corresponds to solutions in Sobolev spaces). Furthermore, the solution enjoys the domain of dependence or finite propagation speed property, with the domain of dependence of a point $x \in X'$ determined by the characteristic cone $\{p(\xi) = 0\}$ at x .

Proof of Theorem 1.4. We shall verify that the modified incompressible Einstein-Navier-Stokes system is of the form (4.1) and satisfies all the conditions given in [6] which we summarized above.

Consider the unknown $V = (g, s, u, \Omega, C) = (v^1, v^2, v^3, v^4, v^5)$ for the system (3.19). Naturally, it is understood that each v^I and each equation in (3.19) represent, respectively, a set of unknowns and a set of equations, but they can be grouped together since they are all of the same form. For instance, for all the ten unknowns g , all the equations take the same form (3.5). We also remark that, as it is standard in the study of the evolution problem in General Relativity, although V and (3.19) are defined in a local coordinate patch, we rely on the aforementioned results [6, 49], given for $\mathbb{R}^n \times [0, T]$ ($n = 3$ in our case), using the finite propagation speed and a standard gluing argument to construct global in space solutions (see below).

One verifies that (3.19) has the form (4.1) upon choosing

$$\begin{aligned} m_1 &= 3, & m_2 &= 2, & m_3 &= 2, & m_4 &= 1, & m_5 &= 2, \\ n_1 &= 1, & n_2 &= 0, & n_3 &= 0, & n_4 &= 0, & n_5 &= 0, \end{aligned} \tag{4.2}$$

where $m_1 = m(v^1) \equiv m(g)$, $m_2 \equiv m(v^2) = m(s)$, $m_3 = m(v^3) \equiv m(u)$, $m_4 = m(v^4) \equiv m(\Omega)$, $m_5 = m(v^5) \equiv m(C)$, $n_1 = n(\text{equation (3.19a)}) \equiv n(\text{equation (3.5)})$, $n_2 = n(\text{equation (3.19b)}) \equiv n(\text{equation (3.6)})$, $n_3 = n(\text{equation (3.19c)}) \equiv n(\text{equation (3.16)})$, $n_4 = n(\text{equation (3.19d)}) \equiv n(\text{equation (3.17)})$, $n_5 = n(\text{equation (3.19e)}) \equiv n(\text{equation (3.18)})$, and letting h_I^J be the differential operator whose matrix (h_I^J) is given by (3.20). Indeed, we list below for each equation J in (3.19), the value of n_J , the highest derivatives of the each unknown entering in the coefficients and

the right-hand side of the equation, and the difference $m_I - n_J$:

$$\begin{aligned}
\text{eq. (3.19a) : } n_1 = 1; \partial g, s, u, C; & \left\{ \begin{array}{l} m(g) - n_1 \equiv m_1 - n_1 = 2, \\ m(s) - n_1 \equiv m_2 - n_1 = 1, \\ m(u) - n_1 \equiv m_3 - n_1 = 1, \\ m(\Omega) - n_1 \equiv m_4 - n_1 = 0, \\ m(C) - n_1 \equiv m_5 - n_1 = 1, \end{array} \right. \\
\text{eq. (3.19b) : } n_2 = 0; \partial^2 g, \partial s, \partial u, \partial C; & \left\{ \begin{array}{l} m(g) - n_2 \equiv m_1 - n_2 = 3, \\ m(s) - n_2 \equiv m_2 - n_2 = 2, \\ m(u) - n_2 \equiv m_3 - n_2 = 2, \\ m(\Omega) - n_2 \equiv m_4 - n_2 = 1, \\ m(C) - n_2 \equiv m_5 - n_2 = 2, \end{array} \right. \\
\text{eq. (3.19c) : } n_3 = 0; \partial^2 g, \partial s, \partial u, \Omega, \partial C; & \left\{ \begin{array}{l} m(g) - n_3 \equiv m_1 - n_3 = 3, \\ m(s) - n_3 \equiv m_2 - n_3 = 2, \\ m(u) - n_3 \equiv m_3 - n_3 = 2, \\ m(\Omega) - n_3 \equiv m_4 - n_3 = 1, \\ m(C) - n_3 \equiv m_5 - n_3 = 2, \end{array} \right. \\
\text{eq. (3.19d) : } n_4 = 0; \partial^2 g, \partial s, \partial u, \Omega, \partial C; & \left\{ \begin{array}{l} m(g) - n_4 \equiv m_1 - n_4 = 3, \\ m(s) - n_4 \equiv m_2 - n_4 = 2, \\ m(u) - n_4 \equiv m_3 - n_4 = 2, \\ m(\Omega) - n_4 \equiv m_4 - n_4 = 1, \\ m(C) - n_4 \equiv m_5 - n_4 = 2, \end{array} \right. \\
\text{eq. (3.19e) : } n_5 = 0; \partial^2 g, \Omega, \partial C; & \left\{ \begin{array}{l} m(g) - n_5 \equiv m_1 - n_5 = 3, \\ m(s) - n_5 \equiv m_2 - n_5 = 2, \\ m(u) - n_5 \equiv m_3 - n_5 = 2, \\ m(\Omega) - n_5 \equiv m_4 - n_5 = 1, \\ m(C) - n_5 \equiv m_5 - n_5 = 2. \end{array} \right.
\end{aligned}$$

As described in [6, 49], the Cauchy data for a system of the form (4.1) consists of the functions v^I , along with their derivatives up to order $m_J - 1$, on the initial surface. The initial data is also required to satisfy some compatibility conditions, which essentially come from requiring that the equations are satisfied on the initial time slice when they take the values of the initial data. In our case, we have to further ensure that the initial data for the system (3.19) is compatible with solutions of the original set of equations, i.e., (1.15) (written in harmonic coordinates), and with (1.2) and (1.10).

The derivation, out of the original initial data, of Cauchy data for (3.19), such that the conditions of the last paragraph are satisfied, is done in similar fashion as in [13], and therefore we shall skip the details. In a nutshell, one uses the equations of motion to derive what the new initial data ought to be. In fact, such a procedure is commonly used in General Relativity when solutions to Einstein equations are found via a different set of equations [8, 10, 14, 17, 26, 31, 30, 29, 34, 32, 33, 55]. We remark, for future reference, that although we are treating Ω , C , and u as independent variables,

and hence we do not know yet that (1.2) and (1.10) hold true, these relation are satisfied by the initial data by the way they are derived; see [13].

Next, we need to compute the characteristic determinant of the system, $\det A(V, \xi)$, where ξ is a co-vector and $A(V, \xi)$ the principal symbol in the direction of ξ . From (3.20),

$$\det A(V, \xi) = \det a_{11}(g, \xi) \det a_{22}(g, u, \xi) \det a_{33}(g, \xi) \det \tilde{A}(g, u, C, \xi), \quad (4.3)$$

where $a_{ij}(\cdot, \xi)$ and $\tilde{A}(g, u, C, \xi)$ are, respectively, the principal symbols of the differential operators a_{ij} and

$$\tilde{A}(g, u, C, \partial) = \begin{pmatrix} a_{44}(g, C) \partial & a_{45}(g, u) \partial^2 \\ a_{54}(g) \partial & a_{55}(g) \partial^2 \end{pmatrix}.$$

From (3.5), (3.6), (3.16), we find,

$$\det a_{11}(g, \xi) = (\xi^\mu \xi_\mu)^{10}, \quad (4.4)$$

$$\det a_{22}(g, u, \xi) = (u^\mu \xi_\mu)^2, \quad (4.5)$$

and

$$\det a_{33}(g, \xi) = (\xi^\nu \xi_\nu)^4, \quad (4.6)$$

where as usual the indices are raised with g , i.e., $\xi^\mu \xi_\mu = g^{\mu\nu} \xi_\nu \xi_\mu$, and $u^\mu \xi_\mu = g^{\mu\nu} u_\nu \xi_\mu$. The powers 10 and 4 in (4.4) and (4.6) come, respectively, from the fact that (3.5) corresponds to ten equations and (3.16) to four equations, whereas the power 2 in (4.5) comes from the double characteristic $u^\alpha u^\beta \xi_\alpha \xi_\beta$ of $u^\alpha u^\beta \partial_{\alpha\beta} s$ in (3.6).

The operator \tilde{A} has a more complicated structure, which requires us to be more explicit. Recalling that Ω has six independent components, the components (Ω, C) in $V = (g, s, u, \Omega, C)$ are

$$(\Omega_{01}, \Omega_{02}, \Omega_{03}, \Omega_{12}, \Omega_{13}, \Omega_{23}, C_0, C_1, C_2, C_3),$$

From (3.17) and (3.18), we then see that \tilde{A} has the following form

$$\begin{bmatrix} C^\mu \partial_\mu & 0 & 0 & 0 & 0 & 0 & -qu^\mu \partial_{\mu 1} & qu^\mu \partial_{\mu 0} & 0 & 0 \\ 0 & C^\mu \partial_\mu & 0 & 0 & 0 & 0 & -qu^\mu \partial_{\mu 2} & 0 & qu^\mu \partial_{\mu 0} & 0 \\ 0 & 0 & C^\mu \partial_\mu & 0 & 0 & 0 & -qu^\mu \partial_{\mu 3} & 0 & 0 & qu^\mu \partial_{\mu 0} \\ 0 & 0 & 0 & C^\mu \partial_\mu & 0 & 0 & 0 & -qu^\mu \partial_{\mu 2} & qu^\mu \partial_{\mu 1} & 0 \\ 0 & 0 & 0 & 0 & C^\mu \partial_\mu & 0 & 0 & -qu^\mu \partial_{\mu 3} & 0 & qu^\mu \partial_{\mu 1} \\ 0 & 0 & 0 & 0 & 0 & C^\mu \partial_\mu & 0 & 0 & -qu^\mu \partial_{\mu 3} & qu^\mu \partial_{\mu 2} \\ g^{\mu 1} \partial_\mu & g^{\mu 2} \partial_\mu & g^{\mu 3} \partial_\mu & 0 & 0 & 0 & g^{\mu\nu} \partial_{\mu\nu} & 0 & 0 & 0 \\ -g^{\mu 0} \partial_\mu & 0 & 0 & g^{\mu 2} \partial_\mu & g^{\mu 3} \partial_\mu & 0 & 0 & g^{\mu\nu} \partial_{\mu\nu} & 0 & 0 \\ 0 & -g^{\mu 0} \partial_\mu & 0 & -g^{\mu 1} \partial_\mu & 0 & g^{\mu 3} \partial_\mu & 0 & 0 & g^{\mu\nu} \partial_{\mu\nu} & 0 \\ 0 & 0 & -g^{\mu 0} \partial_\mu & 0 & -g^{\mu 1} \partial_\mu & -g^{\mu 2} \partial_\mu & 0 & 0 & 0 & g^{\mu\nu} \partial_{\mu\nu} \end{bmatrix}$$

$\tilde{A}(g, u, C, \xi)$ is given by

$$\begin{bmatrix} C^\mu \xi_\mu & 0 & 0 & 0 & 0 & 0 & -qu^\mu \xi_\mu \xi_1 & qu^\mu \xi_\mu \xi_0 & 0 & 0 \\ 0 & C^\mu \xi_\mu & 0 & 0 & 0 & 0 & -qu^\mu \xi_\mu \xi_2 & 0 & qu^\mu \xi_\mu \xi_0 & 0 \\ 0 & 0 & C^\mu \xi_\mu & 0 & 0 & 0 & -qu^\mu \xi_\mu \xi_3 & 0 & 0 & qu^\mu \xi_\mu \xi_0 \\ 0 & 0 & 0 & C^\mu \xi_\mu & 0 & 0 & 0 & -qu^\mu \xi_\mu \xi_2 & qu^\mu \xi_\mu \xi_1 & 0 \\ 0 & 0 & 0 & 0 & C^\mu \xi_\mu & 0 & 0 & -qu^\mu \xi_\mu \xi_3 & 0 & qu^\mu \xi_\mu \xi_1 \\ 0 & 0 & 0 & 0 & 0 & C^\mu \xi_\mu & 0 & 0 & -qu^\mu \xi_\mu \xi_3 & qu^\mu \xi_\mu \xi_2 \\ \xi^1 & \xi^2 & \xi^3 & 0 & 0 & 0 & \xi^\mu \xi_\mu & 0 & 0 & 0 \\ -\xi^0 & 0 & 0 & \xi^2 & \xi^3 & 0 & 0 & \xi^\mu \xi_\mu & 0 & 0 \\ 0 & -\xi^0 & 0 & -\xi^1 & 0 & \xi^3 & 0 & 0 & \xi^\mu \xi_\mu & 0 \\ 0 & 0 & -\xi^0 & 0 & -\xi^1 & -\xi^2 & 0 & 0 & 0 & \xi^\mu \xi_\mu \end{bmatrix}$$

The determinant of the above matrix can be computed, yielding, after much algebra,

$$F^3(F+q)^2(u^\mu \xi_\mu)^6(\xi^\lambda \xi_\lambda)^2 P(\xi), \quad (4.7)$$

where

$$P(\xi) = A\xi_0^4 + B\xi_0^2 + C$$

and we have used that $g_{00} = 1$ and $g_{0i} = 0$. The coefficients A , B , and C are given by

$$A = F + q,$$

$$B = 2F\xi_1\xi^1 + 2q\xi_1\xi^1 + 2F\xi_2\xi^2 + 2q\xi_2\xi^2 + q\xi_3\xi^2 + 2F\xi_3\xi^3 + q\xi_3\xi^3,$$

and

$$\begin{aligned} C = & F(\xi_1\xi^1)^2 + q(\xi_1\xi^1)^2 + 2F\xi_1\xi^2\xi_2\xi^2 + 2q\xi_1\xi^2\xi_2\xi^2 + q\xi_1\xi_3\xi^1\xi^2 + F(\xi_2\xi^2)^2 + q(\xi_2\xi^2)^2 \\ & + q\xi_2\xi_3(\xi_2)^2 + 2F\xi_1\xi_3\xi^1\xi^3 + q\xi_1\xi_3\xi^1\xi^3 + 2F\xi_2\xi_3\xi^2\xi^3 + q\xi_2\xi_3\xi^2\xi^3 + q(\xi_3)^2\xi^2\xi^3 + F(\xi_3\xi^3)^2. \end{aligned}$$

We investigate the roots of $P(\xi)$. We have

$$(\xi_0)^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

We need to verify that the right-hand side is real and non-negative. A long but not difficult computation reveals that

$$B^2 - 4AC = q^2\xi_3^2(\xi^2 - \xi^3)^2,$$

assuring reality. For non-negativity, it suffices to show that $-B - \sqrt{B^2 - 4AC} \geq 0$. For this, assume first that we are working at a point where g equals the Minkowski metric, so that, after some more algebra and recalling our sign convention,

$$-B - \sqrt{B^2 - 4AC} = 2(F+q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + F(\xi_3)^2 + q\xi_3\xi_2 - \sqrt{q^2\xi_3^2(\xi_2 - \xi_3)^2}.$$

It suffices to analyze the case where the term $q\xi_3\xi_2$ gives a non-positive contribution. Thus we can replace $q\xi_3\xi_2$ by $-q\xi_3\xi_2$ and assume that $\xi_2 \geq 0$ and $\xi_3 \geq 0$, in which case the above becomes

$$-B - \sqrt{B^2 - 4AC} = 2(F+q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + F(\xi_3)^2 - q\xi_3\xi_2 - q\xi_3|\xi_2 - \xi_3|.$$

If $\xi_2 \geq \xi_3$, we find that

$$\begin{aligned}
-B - \sqrt{B^2 - 4AC} &= 2(F + q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + F(\xi_3)^2 - q\xi_3\xi_2 - q\xi_3(\xi_2 - \xi_3) \\
&= 2(F + q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + (F + q)(\xi_3)^2 - 2q\xi_2\xi_3 \\
&\geq 2(F + q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + (F + q)(\xi_3)^2 - 2q(\xi_2)^2 \\
&= 2(F + q)(\xi_1)^2 + 2F(\xi_2)^2 + 2(F + q)(\xi_3)^2 \geq 0.
\end{aligned} \tag{4.8}$$

If $\xi_2 \leq \xi_3$:

$$\begin{aligned}
-B - \sqrt{B^2 - 4AC} &= 2(F + q)((\xi_1)^2 + (\xi_2)^2 + \frac{(\xi_3)^2}{2}) + F(\xi_3)^2 - q\xi_3\xi_2 - q\xi_3(\xi_3 - \xi_2) \\
&= 2(F + q)((\xi_1)^2 + (\xi_2)^2) + (F + q)(\xi_3)^2 + F(\xi_3)^2 - q(\xi_3)^2 \\
&= 2(F + q)((\xi_1)^2 + (\xi_2)^2) + 2F(\xi_3)^2 \geq 0.
\end{aligned} \tag{4.9}$$

Therefore, we conclude that $P(\xi)$ factors as the product of two hyperbolic polynomials of degree two, $P(\xi) = P_1(\xi)P_2(\xi)$, at least at a point where the metric equals the Minkowski metric.

Now we consider the general case, i.e., when g does not necessarily equal the Minkowski metric. Consider the initial hypersurface Σ where the Cauchy data is given. We can assume that the coordinate chart on Σ is the neighborhood of a point p such that $g_{\alpha\beta}(p) = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the Minkowski metric. Notice that we can still assure that the same coordinates are harmonic coordinates, since the latter are determined by prescribing the first derivatives of g . Since (4.8) and (4.9) are strict inequalities when $\xi \neq 0$, we see that $-B - \sqrt{B^2 - 4AC} \geq 0$ for points sufficiently near p . Therefore, $P(\xi)$ is the product of two hyperbolic polynomial, as desired (notice that the reality condition previously verified did not use $g_{\alpha\beta}(p) = \eta_{\alpha\beta}$).

Combining (4.3), (4.4), (4.5), (4.6), (4.7), and the above, we conclude that

$$\det A(V, \xi) = F^3(F + q)^3(\xi^\mu \xi_\mu)^{14}(u^\nu \xi_\nu)^6(u^\tau \xi_\tau \xi^\rho \xi_\rho)^2 P_1(\xi) P_2(\xi), \tag{4.10}$$

where it is understood that the above expression is evaluated at the initial data since, as mentioned in the beginning of this section, we need to verify the hyperbolicity conditions of [6] when the unknown V takes the values of the Cauchy data¹³. In particular, also as already mentioned, even though we are treating C and u as independent variables, for the initial data it holds that $C = Fu$, and we used this fact to eliminate C from (4.10).

It is well-known (see e.g. [55]) that the first, second, and third degree polynomials $u^\tau \xi_\tau$, $\xi^\mu \xi_\mu$, and $u^\nu \xi_\nu \xi^\rho \xi_\rho$, are hyperbolic as long as g is a Lorentzian metric and u is time-like, conditions that are to be fulfilled when V takes our initial conditions. Also, $F + q > 0$ because $q > 0$ by hypothesis and $F \geq 1$ by (1.3), which holds for the initial data, as well as the fact that $\epsilon \geq 0$, $p \geq 0$, and $r > 0$. $\det A(V, \xi)$ is, therefore, a product of 24 hyperbolic polynomials, with the highest degree of such polynomials being equal to three. Thus, in the notation employed at the beginning of this section and with indices m_I and n_J given by (4.2), we verify that

$$3 = \max_q \ell_q \geq \max_I m_I - \min_J n_J = 3 - 0 = 3.$$

We also have $\sigma_0 = \frac{24}{24-1} = \frac{24}{23}$.

¹³More precisely, we need to verify the hyperbolicity conditions when V and its derivatives up to an order determined by the compatibility conditions take the values of the Cauchy data [6, 13]. Here, however, the coefficients appearing in the determinant (4.10) do not involve derivatives of V .

The coefficients of the differential operator $A(V, \partial)$ depend polynomially on V , whereas $B(V)$ is a rational function of the functions v^I . The denominator of the rational expressions appearing in $B(V)$ are products of $\sqrt{C^\rho C_\rho}$, $r = r(F, s) \equiv r(\sqrt{C^\rho C_\rho}, s)$, and $\theta = \theta(F, s) \equiv \theta(\sqrt{C^\rho C_\rho}, s)$. Hence, recalling (1.13) and that $F > 0$, the denominators in such rational expressions are, as functions of V , uniformly bounded away from zero (recall that Σ is compact) when V takes the Cauchy data.

We have, therefore, verified all the conditions necessary to apply Choquet-Bruhat's theorem [6] combined with Leray and Ohya's results [49], obtaining a short-in-time solution V to (3.19) with $v^I \in \gamma^{m_I, (\sigma)}(\Sigma \times [0, T])$, for $1 \leq \sigma < \sigma_0$ and some $T > 0$.

It has to be shown that the solution V to (3.19) yields a solution to the original set of equations (1.15). The argument to show this is very similar to the one employed in [13, 56] (see also [55]), thus we just mention the general idea. Consider the incompressible Einstein-Navier-Stokes system written in harmonic coordinates. Pichon [69] has shown that this system can be solved for analytic data (his work treated only the case of an equation of state that does not include entropy, but it is not difficult to see that his procedure generalizes to the case of interest here). By the way the Cauchy data for (3.19) is derived out of the initial data for (1.15), the analytic solution to (1.15) will satisfy the system (3.19) with $C_\alpha = Fu_\alpha$ and $\Omega_{\alpha\beta} = \nabla_\alpha C_\beta - \nabla_\beta C_\alpha$. For the case of initial data in Gevrey spaces, as in Theorem 1.4, we approximate the initial data by analytic Cauchy data, obtaining a sequence $\{(g_j, u_j, r_j, s_j)\}$ of analytic solutions to (1.15), and a corresponding sequence $\{V_j\}$ of analytic solutions to (3.19) that converges to the solution V obtained above. The estimates on solutions derived by Leray and Ohya [49] assure that $\{(g_j, u_j, s_j, r_j)\}$ also converges to a limit $\{(g, u, s, r)\}$ that satisfies the incompressible Einstein-Navier-Stokes system and belongs to the desired Gevrey class. It is well-known that a solution to Einstein's equations in harmonic coordinates yields a solution to the full system if and only if the constraint equations are satisfied, which is the case by hypothesis¹⁴. Finally, we notice that $\pi^{\gamma\beta} \nabla^\alpha T_{\alpha\beta} = 0$ implies

$$0 = u^\alpha u^\gamma \nabla_\alpha u_\gamma = \frac{1}{2} u^\alpha \partial_\alpha (u^\gamma u_\gamma),$$

and therefore u , being unitary at time zero, remains unitary.

The existence of a domain of dependence also follows from the results of [49]. The domain of dependence of the solution is given by the intersection of the interior of the cones determined by the hyperbolic polynomials appearing in the product (4.10). All these cones have a common interior, namely, the interior of the light-cone $\xi^\mu \xi_\mu = g^{\mu\nu} \xi_\mu \xi_\nu \geq 0$. With the domain of dependence at hand, a standard gluing argument produces a solution that is global in space and geometrically unique. This finishes the proof of Theorem 1.4.

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¹⁴Here it is important to remind the reader of the discussion of section 2.1. In particular, the solvability of the constraints under the further requirements imposed by (1.14) is at this point unknown.

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